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Mean Approximation by Polynomials on a Jordan Curve*

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If C is an analytic Jordan curve of the z-plane, studies have been made [1, 5, 8] of the relative inclusion of the classes $H_p(k, \alpha)$ and various classes of degree of approximation on C by polynomials in z, or by polynomials in z and $1/z$. However, no corresponding study other than [3] seems to have been made of the relative properties of approximation by polynomials in z and $1/z$ on the one hand and of the series formed on C by the components (parts containing z or $1/z$ only) of such polynomials; the object of the present note is to make such a study for $p > 1$. Our methods are in part those of Hardy and Littlewood in polynomial approximation, of Quade in proof in detail of some of the Hardy–Littlewood results, and of Zygmund in deepening those results.[1]

If Γ is the unit circumference $|z| = 1$, and if $F(z)$ is a function of class L^p $(p > 1)$ on Γ, then there exist [10, p. 151] two unique functions $f(z)$ and $g(z)$ of respective classes H_p and G_p on Γ such that

$$F(z) \equiv f(z) + g(z) \tag{1}$$

a.e. on Γ; here H_p is the Hardy class of functions $f(z)$, analytic interior to Γ, with

$$\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta$$

bounded for $0 < r < 1$; and G_p is the corresponding class of functions $g(z)$ analytic exterior to Γ, and satisfying $g(\infty) = 0$. Boundary (Fatou) values of $f(z)$ and $g(z)$ in L^p exist a.e. on Γ. As a consequence of inequalities due to M. Riesz, one has also [10, p. 151] the inequalities

$$\|f\|_p, \|g\|_p \leqslant C_p \|F\|_p, \tag{2}$$

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where C_p is a constant depending merely on p. We always suppose $1 < p < \infty$.

Incidentally, inequalities (2) are almost trivial in the special case $p = 2$, for we have the formal developments on Γ

$$F(z) \sim \sum_{0}^{\infty} a_n z^n, \qquad F(z) \sim \sum_{-1}^{-\infty} a_n z^n, \tag{3}$$

and the two sets $\{z, z^2, z^3,...\}$ and $\{z^{-1}, z^{-2},...\}$ are not merely orthogonal sets but are orthogonal to each other on Γ. Then (1) and (2) hold, for we have $\| F \|_2^2 = \| f \|_2^2 + \| g \|_2^2$.

If $f(z) \in H_p$ and if the boundary values (on Γ) possess a k-th derivative which satisfies a p-th mean integrated Lipschitz condition of order α $(0 < \alpha < 1)$ or a p-th mean integrated Zygmund condition $(\alpha = 1)$

$$\int_0^{2\pi} | f^{(k)}(e^{i\theta}) + f^{(k)}(e^{i\theta+2h}) - 2f^{(k)}(e^{i\theta+h})|^p \, d\theta \leqslant A \mid h \mid^p,$$

then we write $f(z) \in H_p(k, \alpha)$ on Γ; here and below the constant A depends only on $f(z)$, k, and p and may change from one inequality to another. These classes $H_p(k, \alpha)$ are by definition [4] invariant under one-to-one conformal transformation of Γ and its interior.

The fundamental theorems on polynomial approximation to such and similar functions are now stated.

THEOREM 1. *If a function $F(z)$ is of class $L^p(k, \alpha)$ on Γ, then there exist polynomials $P_n(z, 1/z)$ of respective degrees n in z and $1/z$ satisfying on Γ*

$$\| F(z) - P_n(z, 1/z) \|_p \leqslant A/n^{k+\alpha}, \qquad 0 < \alpha \leqslant 1,$$

and conversely.

Theorem 1 was formulated by Hardy and Littlewood for $0 < \alpha < 1$, and proved by E. S. Quade; it is due, for the case $\alpha = 1$, to A. Zygmund. An analog [2, 4] is

THEOREM 2. *A necessary and sufficient condition that a function $F(z)$ be of class $H_p(k, \alpha)$ on Γ is that there exist polynomials $p_n(z)$ in z of respective degrees n satisfying on Γ*

$$\| F(z) - p_n(z) \|_p \leqslant A/n^{k+\alpha}, \qquad 0 < \alpha \leqslant 1.$$

The definition of the class $G_p(k, \alpha)$ is analogous to that of $H_p(k, \alpha)$ (the functions must be analytic exterior to Γ and zero at infinity) and can be readily formulated by the reader; the phrase "of class $H_p(k, \alpha)$ or $G_p(k, \alpha)$ on Γ" should not be confusing. The analog of Theorem 2 follows at once.

Under the conditions of Theorem 1 we may write (1) and $P_n(z, 1/z) \equiv p_n(z) + q_n(z)$, where the latter two functions are polynomials in z and $1/z$, respectively, with $q_n(\infty) = 0$. For by use of Cauchy's integral we have

$$g(z) - q_n(z) \equiv \frac{1}{2\pi i} \int_\Gamma \frac{F(t) - P_n(t, 1/t)}{t - z} dt, \qquad z \text{ exterior to } \Gamma,$$

$$| g(\infty) - q_n(\infty)| \leqslant A/n^{k+\alpha};$$

thus, replacement of $g(z)$ by $g(z) - g(\infty)$ and of $q_n(z)$ by $q_n(z) - q_n(\infty)$, with corresponding replacements of $f(z)$ and of $p_n(z)$, leaves unchanged the condition

$$\| F(z) - P_n(z, 1/z)\|_p \leqslant A/n^{k+\alpha}, \qquad z \text{ on } \Gamma.$$

In the proof of Theorems 3 and 4 we suppose such replacements to be made.

Our main theorem (later to be generalized to other Jordan curves) relates Theorems 1 and 2:

THEOREM 3. *If a function $F(z)$ satisfies the conditions of Theorem 1 for $p > 1$, then we may write uniquely $F(z) \equiv f(z) + g(z)$ on Γ, where $f(z)$ is of class $H_p(k, \alpha)$ on Γ, and $g(z)$ is of class $G_p(k, \alpha)$ on Γ. Indeed, we may write for z on Γ,*

$$\|f(z) - p_n(z)\|_p \leqslant A/n^{k+\alpha}, \qquad \| g(z) - q_n(1/z)\|_p \leqslant A/n^{k+\alpha},$$

where $p_n(z)$ and $q_n(z)$ are the respective components of $P_n(z, 1/z)$. These inequalities are equivalent to estimates of the degree of approximation on Γ by special trigonometric polynomials, namely the power series type and power series with negative exponents type.

We return to inequalities (2). The classes H_p and $H_p(k, \alpha)$ are additive, whence by Theorem 1,

$$\|f(z) - p_n(z)\|_p \leqslant C_p \| F(z) - P_n(z, 1/z)\|_p \leqslant A_2 n^{k+\alpha},$$

with a similar inequality for $\| g(z) - q_n(1/z)\|_p$. It now follows from Theorem 2 that $f(z) \in H_p(k, \alpha)$ on Γ, and similarly that $g(z) \in G_p(k, \alpha)$ on Γ. Theorem 3 is established.

Our purpose henceforth is to extend Theorem 3 from the case of the unit circle Γ to that of an arbitrary analytic Jordan curve C. The definitions of the classes H_p and G_p carry over directly (by conformal map) to such a curve C. A function $f(z)$, analytic interior [exterior] to C, is of class $H_p[G_p]$ on C when and only when the integrals $M_p(r)$ are bounded, if the interior [exterior] of C is mapped onto the interior of Γ. This condition is satisfied too if the numbers $M_p(r)$ are bounded for r sufficiently near unity, where $f(z)$ belongs then either to H_p or to G_p. For $f(z) \in G_p$ we require also $f(\infty) = 0$.

THEOREM 4. *Let C be an arbitrary analytic Jordan curve of the z-plane, containing 0 in its interior, and let $F(z) \in L^p(k, \alpha)$ on C. Then we may write $F(z) \equiv f(z) + g(z)$ on C, where $f(z)$ is of class $H_p(k, \alpha)$ on C, and $g(z)$ is of class $G_p(k, \alpha)$ on C.*

Let $w = \phi(z)$ and $z = \psi(w)$ map C and its interior conformally onto $\Gamma : |w| = 1$ and its interior, with $\phi(0) = 0$; suppose too that the analytic Jordan curve C' exterior to C is mapped simultaneously onto a circle Γ' concentric with Γ, so that the closed annulus (C', C) is mapped by $w = \phi(z)$ one-to-one and conformally onto the closed annulus (Γ', Γ). The function $F[\psi(w)]$ is of class $L^p(k, \alpha)$ on Γ; hence, by Theorem 3 there exist functions $F_1(w)$ of class $H_p(k, \alpha)$ on Γ and $F_2(w)$ of class $G_p(k, \alpha)$ on Γ, $(F_2(\infty) = 0)$ such that $F[\psi(w)] \equiv F_1(w) + F_2(w)$ on Γ. The function $F_1(w)$ is transformed into $F_1[\phi(z)]$, analytic interior to C, of class $H_p(k, \alpha)$ on C. The function $F_2(w)$, analytic interior to the annulus (Γ', Γ), is transformed into the function $F_2[\phi(z)]$, analytic interior to the annulus (C', C), of class $L^p(k, \alpha)$ on C. In that latter annulus we may separate $F_2(w) \equiv F_2[\phi(z)]$ into its two components, $F_2[\phi(z)] \equiv \Phi_1(z) + \Phi_2(z)$, where $\Phi_1(z)$ is analytic throughout the interior of C'. The function $\Phi_2(z) \equiv F_2[\phi(z)] - \Phi_1(z)$ is analytic throughout the exterior of C, and has boundary values on C of class $L^p(k, \alpha)$ there. Also, $\Phi_2(z)$ possesses [2, Theorem 5.3] an integral analogous to $M_p(r)$ which is bounded and monotonic, so $\Phi_2(z) \in G_p$ on C, and satisfies $\Phi_2(\infty) = 0$; hence, $\Phi_2(z)$ is of class $G_p(k, \alpha)$ on C.

If we now set $f(z) \equiv F_1[\phi(z)] + \Phi_1(z)$, $g(z) \equiv \Phi_2(z)$, we have $F(z) \equiv f(z) + g(z)$ for z on C, where $f(z)$ is of class $H_p(k, \alpha)$ on C and $g(z)$ is of class $G_p(k, \alpha)$ on C. Theorem 4 is established.

It may be noticed that in the proof of Theorem 4 as given, $g(z) \equiv \Phi_2(z)$ is uniquely determined from $f(z)$ and $\phi(z)$ by $f(z) \equiv f[\psi(w)]$; so *both $f(z)$ and $g(z)$ are uniquely determined in Theorem* 4, since they are uniquely determined in the w-plane of Theorem 4 by virtue of Theorem 3.

It is a consequence of the known extensions [2, 4] of Theorems 1 and 2 from Γ to C and of Theorem 4 that corresponding polynomial expansions of $f(z)$ and $g(z)$ in Theorem 4 exist:

THEOREM 5. *Under the conditions of Theorem* 4, *there exist polynomials $p_n(z)$ and $q_n(z)$ of respective degrees n in z and $1/z$ such that we have on C,*

$$\| f(z) - p_n \|_p \leqslant A/n^{k+\alpha},$$
$$\| g(z) - q_n \|_p \leqslant A/n^{k+\alpha}.$$

For $p = \infty$, Theorem 1 extends to yield the same degree of polynomial approximation if $f(z)$ is of class $H_p(k, \alpha)$ on each of a finite number of mutually exterior analytic Jordan curves. Still broader topological generalizations [9] are known, for $p = \infty$, and the same methods yield, thanks to Theorem 5, the following two theorems, relating to approximation by rational functions [6] and by bounded analytic functions [7]. The proofs (which are left to the reader) would not be possible without Theorem 5.

THEOREM 6. *Let E be a bounded closed set whose boundary J consists of a finite number of mutually disjoint analytic Jordan curves J_j, $J = \bigcup J_j$. Let $f(z)$ be analytic in the interior points of E, continuous almost everywhere on J, and of class $L^p(k, \alpha)$, with $0 < \alpha < 1$, on J. In the extended plane, let the set C complementary to E consist of the mutually disjoint regions $C_1, C_2, \ldots, C_\nu$, and let a point α_j be assigned in each C_j, $j = 1, 2, \ldots, \nu$. We choose integers $m_{nk} > 0$ for $n = \nu, \nu + 1, \ldots$, monotonic nondecreasing with n, such that*

$$\sum_{k=1}^{\nu} m_{nk} = n, \tag{4}$$

where the numbers n/m_{nk} are bounded for all k and n. Then there exist rational functions $R_n(z)$ of respective degrees n whose poles lie in the points α_k counted of respective multiplicities m_{nk} such that we have for z on E (norm on J)

$$\| f(z) - R_n(z) \|_p \leqslant A/n^{k+\alpha}. \tag{5}$$

Theorem 6 remains true for $\alpha = 1$ if the Lipschitz condition on $f^{(k)}(z)$ is replaced by a suitable Zygmund condition. A similar remark applies to Theorems 6, 7 (with the corollary) and 8.

THEOREM 7. *Let E, J, C, α_j, and $f(z)$ satisfy the conditions of Theorem 6. Let D be a region or a finite set of regions of the extended plane containing E but containing neither on its boundary nor interior to it any point α_k. Then there exist functions $R_n(z)$, analytic in D, satisfying*

$$\| f(z) - R_n(z) \|_p \leqslant A/n^{k+\alpha},$$
$$| R_n(z) | \leqslant A_1 R^n \text{ in } D,$$

where R is a constant.

COROLLARY. *Under the conditions of Theorem 6, for every M (> 0) there exists a function $\Phi_M(z)$, analytic in D, such that*

$$| \Phi_M(z) | \leqslant M, \qquad z \text{ in } D,$$
$$\| f(z) - \Phi_M(z) \|_p = m_M,$$

where $(\log M)\, m_M^{1/(k+\alpha)}$ is bounded as $M \to \infty$.

In the direction of a converse to Theorem 6 we have

THEOREM 8. *Under the conditions of Theorem* 6 *on E, J, and C, let points*

$$\alpha_{n1}^{(k)}, \alpha_{n2}^{(k)}, \ldots, \alpha_{nm_{nk}}^{(k)} \tag{6}$$

in C be given, having no limit point on E, with (4) *satisfied,* $k = 1, 2, \ldots, \nu$; $n = \nu, \nu + 1, \ldots; \sum_{k=1}^{\nu} m_{nk} = n$. *Suppose* $R_n(z)$ *is a rational function of degree n with its poles in the n points* (6) *such that* (5) *is valid for some* $f(z)$ *on E. Then* $f(z)$ *is of class* $L^p(k, \alpha)$ *on J.*

Theorems 6 and 8 are essentially invariant under conformal transformation of the extended planes, whereas Theorem 7 and its corollary are of especial interest because they are even invariant under conformal transformation of D.

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